of the form (9) are stable with respect to the slow variable k during a time $T = O(\varepsilon^{-2})$.

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ON THE ASYMPTOTIC STABILITY AND INSTABILITY OF THE ZEROTH SOLUTION OF A NON-AUTONOMOUS SYSTEM*

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A non-autonomous set of differential equations with right side satisfying conditions for the existence of limit sets of differential equations /1, 2/ is considered. Theorems are proved on the limit behaviour of the solutions, on the asymptotic stability and instability of the zeroth solution of such a set in the presence of a Liapunov function with a derivative of constant sign. On the basis of these theorems, sufficient conditions are obtained for the asymptotic stability and instability of the zeroth equilibrium position of a non-autonomous mechanical system. A problem is solved on the asymptotic stabilization of a given three-axis orientation in space for a solid with variable moments of inertia.

1. Consider the following set of differential equations

$$x^{\cdot} = X(t, x) \quad (X(t, 0) \equiv 0) \tag{1.1}$$

where x and X are real n-vectors, the function $X(t, x)$ is defined in the domain $R^+ \times \Gamma$ ($R^+ = [0, +\infty[$, $\Gamma = \{\|x\| \leqslant H < +\infty\}$, $\|x\|$ is a certain norm in R^n) and satisfies conditions (A) from /1/: $X(t, x)$ is measurable in t for fixed x, and is continuous in x for fixed t; for any compact set $\Gamma_1 \subset \Gamma$ two local L_1-functions $h_1(t)$ and $h_2(t)$ exist such that for any $x, y \in \Gamma_1$ and $t \in R^+$

$$\|X(t, x)\| \leqslant h_1(t), \quad \|X(t, x) - X(t, y)\| \leqslant h_2(t)\|x - y\|$$

the function $h_1(t)$ is uniformly continuous in the mean on any segement $[\tau, \tau + 1] \subset R^+$, and the function $h_2(t)$ is bounded in the norm on $[\tau, \tau + 1]$, i.e.

$$\int_E h_1(t)\,dt \leqslant \varepsilon, \quad \int_\tau^{\tau+1} h_2(t)\,dt \leqslant \rho$$

for any measurable set $E \subset [\tau, \tau + 1]$ by a measure less than $\mu = \mu(\varepsilon, \Gamma_1) > 0$, and a certain number $\rho = \rho(\Gamma_1)$.

As is shown in /1/, conditions (A) guarantee the existence of solutions of (1.1), in the Caratheodory sense, and their uniqueness, the compactness (in weak L_1-topology) of the family of functions $\{X(t, x)\}$, satisfying these conditions, particularly the existence of limit functions $\varphi(t, x)$ to $X(t, x)$, the mutual continuity of the solutions of the initial system (1.1), and the solutions of the limit systems

$$x^{\cdot} = \varphi(t, x) \tag{1.2}$$

We note that a special case of conditions (A) is Lipschitz conditions in t and x, which

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is convenient in that for these conditions the limit equations (1.2) retain the structure of the initial equations (1.1) /2/.

We call the function /3/

$$V^{\cdot}(t,\ x)=\lim_{h\to+0}\sup\,(V(t+h,\ x+hX(t,\ x))-V(t,\ x))/h$$

the derivative of the scalar function $V(t, x)$ that satisfies locally the Lipschitz condition in x from Γ uniformly in $t\in R^+$.

We will assume that the scalar non-negative function $W(t, x)$ satisfies the Lipschitz condition in t and x on each compact $[t_0, t_0+\nu]\times\Gamma_1$ $(t_0\geqslant 0, \nu>0, \Gamma_1\subset\Gamma)$. The set of functions $\omega(t, x)$ limiting to $W(t, x)$ will be non-empty, and the convergence of $W_n(t, x)=W(t_n+t, x)$ to $\omega(t, x)$ as $t_n\to+\infty$ will be uniform in each compact mentioned.

We will say that (φ, ω) is the limit pair of functions if $\varphi(t, x)$ and $\omega(t, x)$ are limit functions, respectively, of $X(t, x)$ and $W(t, x)$ for the identical sequence $t_n\to+\infty$.

2. Let $\Omega^+(x(t, t_0, x_0))$ denote the set of limit points of a non-continuable function $x=x(t, t_0, x_0)$ of system (1.1).

Theorem 2.1. We assume that a function $V(t, x)\geqslant 0$ exists whose derivative is permanently negative because of (1.1), $V^{\cdot}(t, x)\leqslant -W(t, x)\leqslant 0$. For each limit pair (φ, ω) we let $M^+((\varphi, \omega))$ be the set formed by non-continuable solutions of the system $x^{\cdot}=\varphi(t, x)$ lying in the set $\{\omega(t, x)=0,\ t\in R^+,\ x\in\Gamma\}$ in its whole interval of definition, and $M_*^+(\{(\varphi, \omega)\})$ is the union of $M^+((\varphi, \omega))$ over all (φ, ω). Then for any solution $x=x(t, t_0, x_0)$ of (1.1), defined in the interval $[t_0, +\infty[$, the set of its limit points satisfies the relation $\Omega^+\cap\Gamma\subset M_*^+(\{(\varphi, \omega)\})$.

Proof. If $\|x(t, t_0, x_0)\|\to+\infty$ or $\Omega^+\subset\partial\Gamma$, then the assertion is evident.

Suppose we have $\Omega^+\cap\Gamma\neq\varnothing$ and $x_0^*\in\Omega^+\cap\Gamma$ for the solution $x=x(t, t_0, x_0)$ so that a sequence $t_n\to+\infty$ exists such that $x(t_n, t_0, x_0)\to x_0^*$. The function $V(t)=V(t, x(t, t_0, x_0))$ has a lower bound and decreases. Consequently, $V(t)\to c$ as $t\to+\infty$. We select a subsequence $\{t_k\}$ from the sequence $\{t_n\}$ so that $X_k(t, x)=X(t_k+t, x)\to\varphi_0(t, x)$, $W_k(t, x)=W(t_k+t, x)\to\omega_0(t, x)$. On the basis of /1/, the sequence $x_k(t)=x(t_k+t, t_0, x_0)$ will converge to $\psi(t)\times(\psi(0)=x_0^*)$ the solution of the system $x^{\cdot}=\varphi_0(t, x)$, uniformly in each interval $[0, \alpha]\subset[0, \beta[$, where $[0, \beta[$ is the interval of definition of $\psi(t)$. From the estimate

$$V(t_k+t)-V(t_k)\leqslant-\int_0^t W_k(\tau, x_k(\tau))\,d\tau\leqslant 0 \tag{2.1}$$

by passing to the limit as $t_k\to+\infty$ and taking account of the uniform convergence of $W_k(t, x)$ to $\omega_0(t, x)$, we obtain

$$c-c\leqslant-\int_0^t\omega_0(\tau, \psi(\tau))\,d\tau\leqslant 0 \tag{2.2}$$

Hence $\omega_0(t, \psi(t))\equiv 0$ for all $t\in[0, \beta[$. Therefore, $x_0^*\in M^+((\varphi_0, \omega_0))$, on the basis of which we conclude that $\Omega^+(x(t, t_0, x_0))\subset M_*^+(\{(\varphi, \omega)\})$.

Remark. Under the conditions of Theorem 2.1 for the non-continuable solution of (1.1) defined in the interval $[t_0, t_1[(t_1<+\infty)$ the set of its limit points $\Omega^+\cap\Gamma$ is contained in a subset, invariant to system (1.1), of the set $\{W(t, x)=0,\ t\in R^+,\ x\in\Gamma\}$, i.e., $\Omega^+\cap\Gamma\subset M^+((X, W))$.

Therefore, for any non-continuable solution of (1.1) the relationship $\Omega^+(x(t, t_0, x_0))\cap\Gamma\subset M_*^+\cup M^+((X, W))$ is satisfied.

Theorem 2.2. Under the conditions of Theorem 2.1 each solution of system (1.1) bounded by the domain $\Gamma_1=\{\|x\|\leqslant H_1<H\}$ approaches unboundedly to the connected compact subset of the set $M_*^+(\{(\varphi, \omega)\})$.

The proof follows from the fact that the set of limit points of such a solution, is connected, compact /3/, and contained in $M_*^+(\{(\varphi, \omega)\})$ on the basis of Theorem 2.1.

The theorems proved develop and extend the appropriate results in /4–8/, described in part in /3/.;

3. *Theorem 3.1.* We assume that: 1) a positive-definite function $V(t, x)\geqslant V_1(\|x\|)$ exists whose derivative $V^{\cdot}(t, x)\leqslant-W(t, x)\leqslant 0$ by virtue of (1.1); 2) for any limit pair (φ, ω), the set $\{\omega(t, x)=0\}$ does not contain solutions of the system $x^{\cdot}=\varphi(t, x)$ except $x=0$. Then the zeroth solution of (1.1) is asymptotically stable with domain of attraction $\Gamma(t)$ such that $\sup\,(V(t, x)$ for $x\in\Gamma(t))\leqslant V_1(H_1)$ $(H_1<H)$.

Proof. It follows from condition 1) of the theorem that $x=0$ is stable and the solutions $x=x(t, t_0, x_0)$, $x_0\in\Gamma(t_0)$ of (1.1) are bounded by the domain $\{\|x\|\leqslant H_1\}$.

Repeating the reasoning for Theorem 2.1, we find that for any solution $x=x(t, t_0, x_0)$, $x_0\in\Gamma(t_0)$

the set of its limit points $\Omega^+(x(t, t_0, x_0))$ is contained in the set $P_*^+(\{(\varphi, \omega)\})$, the union in all pairs (φ, ω) of the subsets $P^+((\varphi, \omega)) \subset \{\omega(t, x) = 0\}$, invariant relative to solutions of the system $x^\cdot = \varphi(t, x)$. But from condition 2) of the theorem $P^+((\varphi, \omega)) \equiv \{x = 0\}$, therefore, $P_*^+(\{(\varphi, \omega)\}) \equiv \{x = 0\}$. This means $\Omega^+(x(t, t_0, x_0)) \equiv \{x = 0\}$, i.e., $\lim x(t, t_0, x_0) = 0$ as $t \to +\infty$.

Theorem 3.2. We assume that: 1) a positive-definite function $V(t, x)$ exists that allows infinitesimally high limits $V_1(\|x\|) \leqslant V(t, x) \leqslant V_2(\|x\|)$, whose derivative $V^\cdot(t, x) \leqslant -W(t, x) \leqslant 0$; 2) at least one limit pair (φ_0, ω_0), exists such that the set $\{\omega_0(t, x) = 0\}$ does not contain solutions of the system $x^\cdot = \varphi_0(t, x)$ except $x = 0$.

Then the zeroth solution of (1.1) is uniformly asymptotically stable in x_0 with domain of attraction $\Gamma_0 = \{\|x\| \leqslant H_0 = V_2^{-1}(V_1(H_1)), H_1 < H\}$.

The proof of this theorem is a modification of the proof of Theorem 2.1 from /2/.

Theorem 3.3. We assume that: 1) a positive-definite function $V(t, x)$ exists that satisfies the Lipschitz conditions in (t, x) (and therefore, allowing infinitesimal high limits)

$$V_1(\|x\|) \leqslant V(t, x) \leqslant V_2(\|x\|) \tag{3.1}$$

whose derivative $V^\cdot(t, x) \leqslant -W(t, x) \leqslant 0$ by virtue of (1.1); 2) for any limit pair of functions (φ, ω) the set $\{\omega(t, x) = 0\}$ does not contain solutions of the system $x^\cdot = \varphi(t, x)$ except $x = 0$. Then the zeroth solution of (1.1) is uniformly asymptotically stable with domain of attraction $\Gamma_0 = \{\|x\| \leqslant H_0 = V_2^{-1}(V_1 \ (H_1))\}$.

Proof. It follows from condition 1) of the theorem that the zeroth solution of (1.1) is uniformly stable and solutions of (1.1) from Γ_0 are bounded by the domain $\Gamma_1 = \{\|x\| \leqslant H_1\}$.

We will show that $x = 0$ is a point of attraction of all the solutions of any limit system from the domain Γ_0.

Let $x = \psi(t)$ $(\psi(t_0) = x_0 \in \Gamma_0)$ be the solution of the limit system $x^\cdot = \varphi_0(t, x)$. By the definition of $\varphi_0(t, x)$, a sequence $t_n \to +\infty$ exists such that $X_n(t, x) = X(t_n + t, x) \to \varphi_0(t, x)$. We select a subsequence $t_k \to +\infty$, such that the subsequences $V_k(t, x) = V(t_k + t, x)$ and $W_k(t, x) = W(t_k + t, x)$ converge uniformly on each compact $[t_0, t_0 + \nu] \times \{\|x\| \leqslant H_2, H_2 > H_1\}$ to $\lambda_0(t, x)$ and $\omega_0(t, x)$ respectively. By virtue of (3.1), we have

$$V_1(\|x\|) \leqslant \lambda_0(t, x) \leqslant V_2(\|x\|) \tag{3.2}$$

Consider the sequence of solutions $x = x_k(t)$ $(t \geqslant t_0)$ of the systems of equations $x^\cdot = X_k(t, x)$ that satisfy the initial conditions $x_k(t_0) = x_0$. From the convergence $X_k(t, x) \to \varphi_0(t, x)$ and the condition $x_k(t_0) = x_0$ we have that the $x_k(t)$ will converge uniformly in each interval $[t_0, t_0 + \nu]$ to $\psi(t)$. The functions $x_k(t)$ will simultaneously be solutions of the initial system (1.1) with the initial conditions $x(t_k + t) = x_0$. Consequently, from condition 1) for $t \geqslant t_0$ we have the estimate

$$V_k(t, x_k(t)) - V_k(t_0, x_0) \leqslant -\int_0^t W_k(\tau, x_k(\tau))\, d\tau$$

from which, passing to the limit as $t_k \to +\infty$, we have

$$\lambda_0(t, \psi(t)) - \lambda_0(t_0, x_0) \leqslant -\int_0^t \omega_0(\tau, \psi(\tau))\, d\tau \leqslant 0$$

Hence, also from (3.2) we conclude that the zeroth solution of the system $x^\cdot = \varphi_0(t, x)$ is stable and its solutions from Γ_0 are limited to the domain Γ_1. The system of equations limiting to $x^\cdot = \varphi_0(t, x)$ will be the limit also to (1.1) in the same way as functions that are the limit to $\omega_0(t, x)$ will be the limit to $W(t, x)$ also. Hence, according to condition 2) of the theorem, if (φ_1, ω_1) is the limit pair to (φ_0, ω_0), then the set $\{\omega_1(t, x) = 0\}$ does not contain solutions of the system $x^\cdot = \varphi_1(t, x)$ except $x = 0$. On the basis of Theorem 3.1 we conclude that the zeroth solution of the system $x^\cdot = \varphi_0(t, x)$ is asymptotically stable with domain of attraction Γ_0.

The uniform stability of the zeroth solution of (1.1) and the fact that Γ_0 is the domain of attraction of the point $x = 0$ of solutions of any limit system (1.2) imply the uniform asymptotic stability of the zeroth solution of (1.1) with the domain of attraction Γ_0 /8/.

Theorem 3.4. We assume that: 1) a function $V(t, x)$ exists that allows infinitesimal high limits and takes positive values for a certain $t = t_0 \geqslant 0$ in any small neighbourhood $x = 0$, whose derivative $V^\cdot(t, x) \geqslant W(t, x) \geqslant 0$ by virtue of (1.1); 2) a limit pair of functions (φ_0, ω_0) exists such that the set $\{\omega_0(t, x) = 0\}$ contains no solutions of the system $x^\cdot = \varphi_0(t, x)$ except $x = 0$. Then the zeroth solution of (1.1) is unstable.

For any arbitrary sequence $t_n \to +\infty$ and a number c we denote by $N(t, c)$ the set of points x of the domain Γ for each of which a subsequence $t_k \to +\infty$ exists such that $\lim V(t_k + t, x) = c$ as $t_k \to +\infty$.

Theorem 3.5. We assume that: 1) a function $V(t,x)$ exists which takes positive values for a certain $t=t_0\geqslant 0$ in any small neighbourhood of $x=0$, which is bounded in the domain $V(t,x)\geqslant 0$, whose derivative $V^{\cdot}(t,x)\geqslant W(t,x)\geqslant 0$ by virtue of (1.1): 2) a sequence $t_n\to+\infty$ exists for which the limit set $N(t,c)$ and the limit pair (φ_0,ω_0) are such that for any $c>0$ the set $N(t,c)\bigcap\{\omega_0(t,x)=0\}$ contains no solutions of the system $x^{\cdot}=\varphi_0(t,x)$. Then the zeroth solution of (1.1) is unstable.

The proofs of Theorems 3.4 and 3.5 are modifications of the proofs of Theorems 3.2 and 3.3.

Remark. The conditions imposed on the right side, $X(t,x)$, of (1.1) can be weakened to conditions for the existence of limit systems of integral equations to (1.1) /9/. Conditions on the function $W(t,x)$ can be weakened in an analogous way.

Theorem 3.1 — 3.5 generalize the theorems on asymptotic stability and instability: for non-autonomous systems with sign-definite derivative; autonomous systems and non-autonomous systems with periodic right side in the presence of Liapunov functions with sign-constant derivative /10—13/. It can be show that the conditions of theorems on the asymptotic stability and instability with two Liapunov functions /14, 15/ with respect to the auxiliary Liapunov function are sufficient for compliance with condition 2) of Theorems 3.1—3.5.

Example. Consider the motion of a solid of variable mass having a fixed point and kinetic symmetry when conserving the principal directions, in a homogeneous gravity field under the effect of resistive forces of the medium

$$\begin{aligned}&Ap^{\cdot}+(C-A)qr=mgz\gamma_2-ap+M_x, && \gamma_1^{\cdot}=r\gamma_2-q\gamma_3\\&Aq^{\cdot}+(A-C)pr=-mgz\gamma_1-bq+M_y, && \gamma_2^{\cdot}=p\gamma_3-r\gamma_1\\&Cr^{\cdot}=M_z, && \gamma_3^{\cdot}=q\gamma_1-p\gamma_2\end{aligned}\tag{3.3}$$

We assume that the components of the moments of the reactive forces M_x and M_y are zero, the resultant moment M_z defines r as a bounded function of time $r=r(t,t_0,r_0)$, the moments of inertia $A(t)$ and $C(t)$, the body mass $m(t)$ and its coordinates $z(t)$, the coefficients of the moments of the resistive forces $a(t)$ and $b(t)$, are bounded and satisfy the conditions

$$\begin{aligned}&A(t)\geqslant A_0>0,\ C(t)\geqslant C_0>0,\ m(t)\geqslant m_0>0\\&z(t)\leqslant z_0<0,\ \mu_1(t)=(2a-A^{\cdot})mgz+A(mgz)^{\cdot}\leqslant-\mu_0\\&\mu_2(t)=(2b-A^{\cdot})mgz+A(mgz)^{\cdot}\leqslant-\mu_0<0\end{aligned}\tag{3.4}$$

Then the equations of motion allow non-uniform rotation around the vertical axis of symmetry

$$p=q=0,\quad r=r(t,\ t_0,\ r_0),\quad \gamma_1=\gamma_2=0,\quad \gamma_3=1\tag{3.5}$$

For the derivative of the function

$$V=-A(p^2+q^2)/(mgz)+\gamma_1^2+\gamma_2^2+(1-\sqrt{1-\gamma_1^2-\gamma_2^2})^2$$

we have $V^{\cdot}\leqslant -W(p,q)\equiv-2\mu_0(p^2+q^2)\leqslant 0$ by virtue of (3.3).

The equations that are limiting to the first two equations (3.3), solved for p and q exist and have the form

$$\begin{aligned}&p^{\cdot}=h_1(t)qr+h_2(t)\gamma_2+h_3(t)p\\&q^{\cdot}=-h_1(t)pr-h_2(t)\gamma_1+h_4(t)q\quad(h_2(t)\leqslant-h_0<0)\end{aligned}$$

We find from these equations that the solution of the limit equations to (3.3), that lie in the set $\{W(p,q)=0\}\equiv\{p=q=0\}$ are only the solutions $p=q=0,\ \gamma_1=\gamma_2=0,\ \gamma_3=1$. Hence, on the basis of Theorem 3.3., we conclude that under conditions (3.4) the motion (3.5) is uniformly asymptotically stable in $p,q,\gamma_1,\gamma_2,\gamma_3$. It can be shown that for $z(t)\geqslant z_0>0,\ \mu_1(t)\geqslant\mu_0,\ \mu_2(t)\geqslant\mu_0>0$ the motion (3.5) will be unstable on the basis of Theorem 3.4.

We note that an analogous problem for a body of constant mass was solved in /14/ by using two Liapunov functions, different problems on the stability of rotation of a variable-mass body were first examined in /15/.

4. We consider a mechanical system with time-dependent constraints described by the Lagrange equations

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\partial L}{\partial q^{\cdot}}\right)-\frac{\partial L}{\partial q}=Q\\&q^T=(q_1,q_2,\ldots,q_n),\ L=L_2+L_1+L_0\\&L_2={}^1/_2(q^{\cdot})^TA(t,q)q^{\cdot},\ L_1=B^T(q)q^{\cdot},\ L_0=L_0(t,q)\end{aligned}\tag{4.1}$$

$Q=Q(t;\ q,\ q^{\cdot})$ is the resultant of the generalized gyroscopic and dissipative forces, $Q^T\cdot q^{\cdot}\leqslant 0$; $\partial L/\partial q\equiv 0,\ Q\equiv 0$ for $q^{\cdot}=q=0$ so that the system has a zeroth equilibrium position

$$q^{\cdot}=q=0\tag{4.2}$$

We assume $L_0(t,0)\equiv 0,\ \partial L/\partial t\geqslant 0$, which is satisfied if for fixed $q^{\cdot}$ and q the system kinetic energy is a non-decreasing, and the potential energy a non-increasing function of t. Then for the derivative of the function $L_2 - L_0$ we have

$$(L_2 - L_0)^{\cdot} = -\partial L/\partial t + Q^T \cdot q^{\cdot} \leqslant Q^T \cdot q^{\cdot} \tag{4.3}$$

We also assume that the quantities $A(t, q)$, $\partial A/\partial T$, $\partial A/\partial q$, $\partial B/\partial q, \partial L/\partial q$, Q are bounded and satisfy the Lipschitz condition in all their variables. Then the limit systems of equations to (4.1) exist and have the form

$$A_*^T q^{\cdot\cdot} + \{(q^{\cdot})^T C_* q^{\cdot}\} + \{D_*^T q^{\cdot}\} + F_* = Q_* \tag{4.4}$$

where $\{(q^{\cdot})^T C_* q^{\cdot}\}$, $\{D_*^T q^{\cdot}\}$ are sets of n quadratic and n linear forms, respectively, the elements of the matrices A_*, $\{C_*\}$, $\{D_*\}$, F_*, Q_* are limiting for corresponding elements from (4.1), in particular

$$F_*(t, q) = \lim_{t_n \to +\infty} \frac{\partial L_0}{\partial q}(t_n + t, q) \tag{4.5}$$

Theorem 4.1. We assume that $L_0(t, q) \leqslant 0$, the dissipative forces are partial dissipation forces, $Q^T \cdot q^{\cdot} \leqslant -a(\| q^{\cdot} \|_k) \times$ ($a(a) > 0$ for $a \neq 0$, $\|q\|_k$ is the norm in R^k in the first k coordinates). Then each bounded motion (4.1) approaches the connected subset of the set $M_*^+ \subset \{q_1^{\cdot} = q_2^{\cdot} = \ldots = q_k^{\cdot} = 0\}$ invariantly with respect to the solution of the limit systems (4.4). If the dissipative forces are forces of total dissipation $Q^T \cdot q^{\cdot} \leqslant -a(\| q^{\cdot} \|)$, then each bounded motion (4.1) approaches the connected subset of the sets of equilibrium positions of all systems (4.4) without limit, i.e., the sets of points q defined by the equalities

$$\lim_{t_n \to +\infty} \frac{\partial L}{\partial q}(t_n + t, q) \equiv 0 \quad (0 \leqslant t < +\infty)$$

On the basis of Theorem 2.2 the proofs follow from relations (4.3), the structure of the limits systems (4.4) to (4.1) and equations (4.5).

Theorem 4.2. We assume that: 1) the function $V = -L_0(t, q)$ is positive-definite; 2) the equilibrium position (4.2) is a non-degenerate isolated position, i.e., $\| \partial L_0/\partial q \| \geqslant f_0(\| q^{\cdot} \|)$ $(f_0(a) = 0 \Leftrightarrow a = 0)$; 3) dissipative forces are forces of total dissipation $Q^T \cdot q^{\cdot} \leqslant -a(\| q^{\cdot} \|)$.

Then the equilibrium position (4.2) is uniformly asymptotically stable.

Proof. Because of the boundedness of $A(t, q)$, $\partial L_0/\partial q$ and condition 1) of the theorem, the function $L_2 - L_0$ is positive-definite, allows of infinitesimal high limits in $q^{\cdot}$ and q, and by virtue of (4.3) there will be $(L_2 - L_0)^{\cdot} \leqslant -a(\| q^{\cdot} \|) \leqslant 0$.

From the structure of the limit system (4.4) we have that its every solution lying in the set $\{a(\| q^{\cdot} \|) = 0\} \equiv \{q_1^{\cdot} = q_2^{\cdot} = \ldots = q_n^{\cdot} = 0\}$ is the solution $q = \text{const}$, defined by the equalities $F_*(t, q) \equiv 0$. But it follows from (4.5) and condition 2) of the theorem that $F_*(t, q) \equiv 0 \Leftrightarrow q \equiv 0$, i.e., that solution can only be zero. We have the result required on the basis of Theorem 3.3.

The following result can be obtained by a modification of the proof executed on the basis of Theorems 3.3. and 3.2.

Theorem 4.3. Under conditions 1) and 2) of the previous theorem let us also have 3) $Q^T \cdot q^{\cdot} \leqslant -\beta(t)\, a(\| q^{\cdot} \|) \leqslant 0$, where $\beta(t) \geqslant 0$, $\beta(t) \geqslant \beta_0 > 0$ for $t \in [t_n, t_n + \nu]$ for a certain sequence $t_n \to +\infty$ such that $t_{n+1} - t_n \leqslant \rho = \text{const}$, and a certain number $\nu > 0$. Then (4.2) is uniformly asymptotically stable. If $\beta(t) \geqslant 0$, $\beta(t) \geqslant \beta_0 > 0$ for $t \in [t_n, t_n + \nu]$ for a certain divergent sequence $t_n \to +\infty$ (i.e., the condition $t_{n+1} - t_n \leqslant \rho$ is not satisfied) and $\nu > 0$, then (4.2) is uniformly asymptotically stable in $(q_0, q_0^{\cdot})$.

We note that this result cannot be obtained on the basis of theorems from /2/.

Theorem 4.4. We assume that: 1) the function $L_0(t, q)$ has no maximum at the point $q = 0$ for a certain $t = t_0 \geqslant 0$; 2) the equilibrium position $q = 0$ is a non-degenerate isolated position, i.e., $\| \partial L_0/\partial q \| \geqslant f_0(\| q \|) \geqslant 0$ $(f_0(a) = 0 \Leftrightarrow a = 0)$; 3) the dissipative forces are such that $Q^T \cdot q^{\cdot} \leqslant -\beta(t)\, a(\| q^{\cdot} \|)$ $(\beta(t) \geqslant 0,\ \beta(t) \geqslant \beta_0 > 0$ for $t \in [t_n, t_n + \nu],\ t_n \to +\infty,\ \nu > 0)$.

Then the equilibrium position (4.2) is unstable.

Theorem 4.5. We assume that: 1) the function $L_0(t, q)$ has no maximum at the point $q = 0$ for a certain $t = t_0 \geqslant 0$; 2) a sequence $t_n \to +\infty$ and a number $\nu > 0$ exist for which for any number $\varepsilon > 0$ a $\delta = \delta(\varepsilon) > 0$ exists such that for all $t \in [t_n, t_n + \nu]$ in the set $\{L_0(t, q) = \varepsilon\}$ the inequality $\| \partial L_0/\partial q \| \geqslant \delta$; 3) the dissipative forces are such that $Q^T \cdot q^{\cdot} \leqslant -\beta(t)\, a(\| q^{\cdot} \|)$ for $t \in [t_n, t_n + \nu]$.

Then (4.2) is unstable.

The proofs of Theorems 4.4 and 4.5 follow from (4.3)–(4.5) and Theorems 3.4 and 3.5.

Remark. Theorems 4.2–4.5 can be extended to the case of dissipative forces with partial dissipation. For instance, Theorem 4.2 remains valid if the following conditions are satisfied instead of 2) and 3): 2) $Q^T \cdot q^{\cdot} \leqslant -a(\| q^{\cdot} \|_k)$ and 3) there are no solutions of any limit system (4.2) in the set $\{q_1^{\cdot} = q_2^{\cdot} = \ldots = q_k^{\cdot} = 0\}$ except $q^{\cdot} = q = 0$.

The theorems proved generalize the results for autonomous /11, 16–18/ and non-autonomous /3, 14/ mechanical systems obtained by using several Liapunov functions.

5. Consider the problem of synthesizing the control moment assuring the asymptotic stability of a given triaxial orientation of a solid with variable moments of inertia.

Let $O_1\xi\eta\zeta$ be the inertial, and $Oxyz$ the rigidly connected coordinate systems of a solid body. The rotational motion of the body can be described by the Euler dynamical equations

$$(I\omega)^{\cdot} + \omega \times I\omega = M,\ \omega = (\omega_x, \omega_y, \omega_z),\ M = (M_x, M_y, M_z) \tag{5.1}$$

($I(t)$ is the inertia tensor in the $Oxyz$ axes, defined by a bounded, positive-definite matrix) and the kinematic equations in Rodrigue-Hamilton parameters /19/

$$2\Lambda^{\cdot} = \Lambda \circ \omega,\ \Lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \tag{5.2}$$

When the bases $O_1\xi\eta\zeta$ and $Oxyz$ coincide, we have $\Lambda = (1, 0, 0, 0)$.

The problem of synthesizing the control moment assuring uniform asymptotic stability of the equilibrium position $\omega = 0,\ \Lambda = (1, 0, 0, 0)$ is solved in the form

$$M = -R(t)\,\omega - a\overline{\lambda},\ \overline{\lambda}^T = (\lambda_1, \lambda_2, \lambda_3),\ a > 0 \tag{5.3}$$

Here $R(t)$ is a bounded matrix selected from the condition that $2R(t) + I^{\cdot}(t)$ is a positive-definite matrix.

The function
$$V = \omega^T I(t)\,\omega + 2\alpha\left((1 - \lambda_0)^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)$$

is positive-definite, allows infinitesimal high limits, and has a derivative $V^{\cdot} = -\omega^T(2R + I^{\cdot})\,\omega \leqslant -\beta_0\omega^2\ (\beta_0 > 0)$. because of (5.1)–(5.3). The limit equations to (5.1) and (5.3), solved for ω, will have the form

$$\omega^{\cdot} = \{\omega^T A_*(t)\,\omega\} + \{\omega^T B_*(t)\} - \alpha C_*(t)\,\overline{\lambda}$$

where $\{\omega^T A_*\omega\}, \{\omega^T B_*\}$ are quadratic and linear forms in $\omega_x, \omega_y, \omega_z$, $\det(|C_*|) \geqslant \gamma > 0$. The equilibrium position $\omega = 0,\ \Lambda = (\pm 1, 0, 0, 0)$ are the unique solutions of these equations and (5.2) in the set $\{\omega_x = \omega_y = \omega_z = 0\}$. Hence, on the basis of Theorem 3.3, any motion of the body under the influence of the control (5.3) will approach unboundedly to one of the equilibrium positions $\omega = 0,\ \Lambda = (\pm 1, 0, 0, 0)$. The problem of synthesizing the moment assuring uniform asymptotic stability of the position $\omega = 0,\ \Lambda = (\pm 1, 0, 0, 0)$, thereby extending the results of /20/ to a body with variable moments of inertia, can also be solved by a method analogous to that given here.

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THE MOTION OF A HEAVY SYMMETRICAL BODY WITH FLEXIBLE RODS ABOUT A FIXED POINT *

V.G. VIL'KE

The motion of a symmetrical solid about its centre of mass is considered in the case, when four mutually orthogonal flexible rods are fixed to it in the equatorial plane of the body ellipsoid of inertia. The deformations of rods is defined by the linear theory of the bending of thin viscoelastic rods, and lead to the evolution of the motion of the solid, i.e. the solid approaches steady rotation about the vertical. the approximate equations in Andoyer variables that define the system evolution are obtained by the method of averaging. The stability of the steady rotations obtained is investigated.

The stability of steady rotations of a solid with a single fixed point and with flexible rods attached to it was investigated in /1, 2/. It was shown in /3/ that the longitudinal deformations of elastic rods fixed to a heavy symmetrical solid rotating about a fixed point results in the body approaching a steady rotation about the vertical axis. In that paper an approximate equation was also obtained, which defined the evolution of motion in terms of the Andoyer variables by the method of averaging.

Let $A_1 = B_1 \neq C_1$, where (A_1, B_1, C_1 are the principal central moments of inertia of the solid about the point O (the centre of mass of the body), and let two paris of elastic rods be positioned along the principal axes of the ellipsoid of inertia Ox_1 and Ox_2. Using the linear theory of the bending of thin rectilinear rods, we determine the radius vector of a point of the rod in the system of coordinates $Ox_1x_2x_3$ in the form

$$\mathbf{R}_1 = s\mathbf{e}_1 + \mathbf{u}_1 = s\mathbf{e}_1 + u_{12}(s,t)\,\mathbf{e}_2 + u_{13}(s,t)\,\mathbf{e}_3$$
$$\mathbf{R}_2 = s\mathbf{e}_2 + \mathbf{u}_2 = u_{21}(s,t)\,\mathbf{e}_1 + s\mathbf{e}_2 + u_{23}(s,t)\,\mathbf{e}_3$$
$$s \in K = [-b, a] \cup [a, b]$$

The kinetic energy and angular momentum of the system are defined by the relations

$$T = \frac{1}{2}(J_1\boldsymbol{\omega}, \boldsymbol{\omega}) + \frac{1}{2}\sum_{i=1}^{2}\int_K [(\boldsymbol{\omega} \times \mathbf{R}_i) + \mathbf{R}_i^{\cdot}]^2 \rho\, ds$$

$$\mathbf{G} = J_1\boldsymbol{\omega} + \sum_{i=1}^{2}\int_K [\mathbf{R}_i \times (\boldsymbol{\omega} \times \mathbf{R}_i + \mathbf{R}_i^{\cdot})]\,\rho\, ds$$

where $\boldsymbol{\omega}(\omega_1, \omega_2, \omega_3{}^*)$ is the angular velocity of rotation of the body, J_1 is the inertia tensor of the body, and ρ is the linear density of the rod material, which is assumed homogeneous. The angular velocity and the inertia tensor are considered in the moving system of coordinates $Ox_1x_2x_3$.

The position of the moving coordinate system relative to the fixed system $O\xi_1\xi_2\xi_3$ (the axis $O\xi_3$ is vertical) is defined by Euler's angles. The generalized momenta and Routh's functional are defined by the relations

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